THE ALGORITHM FOR THE GRAM-SCHMIDT PROCESS

Suppose that $u_1, \cdots, u_n$ is a basis for an n-dimensional inner product space V with inner product $(\cdot, \cdot)$. Then the expression for the Gram-Schmidt process produces an orthogonal basis $\{v_1, \cdots, v_n\}$ for V as follows:

Step 1. $v_1 = u_1$.

Step 2. $v_2 = u_2 - \frac{(u_2, v_1)}{\|v_1\|^2} v_1$.

Step 3. $v_3 = u_3 - \frac{(u_3, v_1)}{\|v_1\|^2} v_1 - \frac{(u_3, v_2)}{\|v_2\|^2} v_2$.

$\vdots$

If we want to further obtain an orthonormal basis $\{w_1, \cdots, w_n\}$, we only need to normalize all v_i:

Step 4. $w_i = \frac{v_i}{\|v_i\|}$ for all $i = 1, \cdots, n$.

Put $V_i = \text{span}\{v_1, \cdots, v_i\}$. Then the Gram-Schmidt process can be simplified as follows:

Step 1. $v_1 = u_1$.

Step 2. $v_2 = u_2 - \text{proj}_{V_1} u_2$.

Step 3. $v_3 = u_3 - \text{proj}_{V_2} u_3$.

$\vdots$

Here $\text{proj}_{V_i} u$ is the orthogonal projection of the vector u onto the subspace V_i.